LEAST MODELS OF SECOND-ORDER SET THEORIES

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Abstract. The main theorems of this paper are (1) there is no least transitive model of Kelley–Morse set theory KM and (2) there is a least β-model—that is, a transitive model which is correct about which of its classes are well-founded—of Gödel–Bernays set theory GBC + Elementary Transfinite Recursion. Along the way I characterize when a countable model of ZFC has a least GBC-realization and show that no countable model of ZFC has a least KM-realization. I also show that fragments of Elementary Transfinite Recursion have least β-models and, for sufficiently weak fragments, least transitive models. These fragments can be separated from each other and from the full principle of Elementary Transfinite Recursion by consistency strength. The main question left unanswered by this article is whether there is a least transitive model of GBC + Elementary Transfinite Recursion.

1. Introduction

The jumping-off point for this article is the following well-known theorem asserting that there is a certain kind of least model of ZFC.

Theorem (Shepherdson [She53], Cohen [Coh63])**.** *If there is a transitive model of* ZFC *then there is a least transitive model of* ZFC*. That is, there is a transitive model M of* ZFC *so that if N is any transitive model of* ZFC *then $M \subseteq N$.*

From a modern perspective, this result is not difficult to prove. This minimal model is L_α where α is the least height of a transitive model of ZFC. The absoluteness of L gives that L_α is contained in every transitive model of ZFC.

This proof can be generalized to get analogous results for many strengthenings of ZFC. If T is an extension of ZFC which has a transitive model and is absolute to L, meaning that $M \models T$ implies $L^M \models T$, then T has a least transitive model. This gives least transitive models of ZFC plus small large cardinals. There is, for instance, a least transitive model of ZFC + there is a Mahlo cardinal. It is not difficult to see, however, that this cannot be extended too far up the large cardinal hierarchy.

Theorem. *There is no least transitive model of any T extending* ZFC *which proves there is a measurable cardinal.*

Proof. Suppose otherwise that N were such a model. Let $j : N \to M$ be the elementary embedding from the measurable cardinal. By elementarity, M believes there is a measurable cardinal. Therefore by the leastness of N we have $M \supseteq N$, violating that the measure giving the embedding is not in M. □

In contrast, results from inner model theory show that if there is a transitive model which thinks the ordinal κ is measurable, then there is a least such model. This extends to strong cardinals, Woodin cardinals, etc.

A second direction the Shepherdson–Cohen result could be generalized is from first-order set theory to second-order set theory. Given some second-order set theory, does it have a least transitive model? A natural first theory to look at here is GBC, Gödel–Bernays set theory with global choice. Implicit in Shepherdson's argument is the answer to this question: there is a least transitive model of GBC.

The main result of this article is that this does not happen with Kelley–Morse set theory, the second natural theory to consider.

Main Theorem 1. *There is no least transitive model of* KM.

The third theory I consider is ETR, intermediate in strength between GBC and KM. This theory is axiomatized by the axioms of GBC plus the Elementary Transfinite Recursion schema. While this article does not answer the question of whether there is a least transitive model of ETR, I do answer the question for β-models of ETR. A transitive model of second-order set theory is a β-model if it is correct about well-foundedness for class relations. This gives a more restricted class of models.

Main Theorem 2. *There is a least β-model of* ETR*, if there is any β-model of* ETR.

Combined with the folklore results that GBC and KM have least β-models, we have the following state of affairs

	least transitive model?	least β-model?
GBC	Y	Y
ETR	?	Y
KM	N	Y

Left open by this article is filling in the question mark in the table: does ETR have a least transitive model?

In the literature the standard terminology is to refer to minimal models rather than, as I do, least models—for instance, Cohen's article [Coh63] is titled "A minimal model for set theory". This terminology is perhaps a misnomer; these models are not only minimal in the order-theoretic sense—they have no smaller submodels—but are also least—they are contained in every model. I depart from the standard language because this article considers both least models and minimal—in the order-theoretic sense—models and I wish for my word choice to differentiate between the two phenomena.

The structure of this article is as follows. I first discuss some preliminary definitions and results. This is followed by a section recalling some facts from admissible set theory, which will be used to prove the results about KM. The remaining three sections are each devoted to one of the major second-order set theories under consideration. These three sections are largely independent. Section 4 is about GBC. I characterize when a countable model of ZFC has a least GBC-realization and show that any countable model of GBC has minimal extensions. Section 5 is about ETR and some weak variants of ETR. I prove main theorem 2 and show that there are least transitive models of certain fragments of ETR. The section ends by separating these fragments of ETR. Section 6 is about KM. I prove main theorem 1 and get as a corollary that no countable model of ZFC has a least KM-realization. Finally, I conclude with a brief discussion of some related theorems about least models of KM that have appeared in the literature.

2. Preliminaries

The second-order set theories considered in this article are formulated so that their models consist of two parts, the sets of or first-order part of the model and the classes of or second-order part of the model. All second-order set theories considered in this paper will include an axiom asserting extensionality holds for classes. This ensures that any model of these theories is isomorphic to one whose second-order part consists of subsets of the first-order part. Without loss we will consider only such models. Formally, models of second-order set theories are of the form $(M, \mathcal{X}, \in_1, \in_2)$ where M is the first-order part, $\mathcal{X}$ is the second-order part, $\in_1$ is the set-set membership relation for the model, and $\in_2$ is the set-class membership relation for the model. For simplicity, I will write $(M, \mathcal{X})$, suppressing writing the membership relations. I will use $\in^M$, or simply $\in$ if it is unambiguous, to refer to both the set-set and the set-class membership relations of M.

A model $(M, \mathcal{X})$ of second-order set theory is *transitive* if both its set-set and set-class membership relations are the true $\in$. For the theories considered in this article, this is equivalent to asking for M to be transitive or asking for $\mathcal{X}$ to be transitive.

Definition 3. Given a second-order set theory T, the *least transitive model of* T, provided it exists, is a transitive model $(M, \mathcal{X})$ of T so that if $(N, \mathcal{Y})$ is any transitive model of T then $M \subseteq N$ and $\mathcal{X} \subseteq \mathcal{Y}$. For the theories we consider, this is equivalent to requiring that $\mathcal{X} \subseteq \mathcal{Y}$.

If M is a first-order model of set theory and T is a second-order set theory, say that $\mathcal{X} \subseteq \mathcal{P}(M)$ is a *T-realization for M* if $(M, \mathcal{X}) \models T$. If there is some T-realization for M then we say that M is *T-realizable*. The *least T-realization for M*, provided it exists, is $\mathcal{X} \subseteq \mathcal{P}(M)$ so that $(M, \mathcal{X}) \models T$ and for any $\mathcal{Y} \subseteq \mathcal{P}(M)$ if $(M, \mathcal{Y}) \models T$ then $\mathcal{X} \subseteq \mathcal{Y}$. That is, $\mathcal{X}$ is a least T-realization for M if it is the least element in the poset consisting of all T-realizations for M ordered by the subset relation.

A transitive model $(M, \mathcal{X})$ of second-order set theory is a *β-model* if it is correct about well-foundedness for class relations. That is, if $R \in \mathcal{X}$ and $(M, \mathcal{X}) \models$ "R is well-founded" then R really is well-founded. Not every transitive model is a β-model, even if we restrict to models satisfying a strong second-order set theory; see corollary 43. This is in contrast to the well-known fact that well-foundedness is absolute for transitive models of ZFC. For models of ZFC, there are two characterizations of well-foundedness: a Π_1 characterization, namely that every nonempty subset of $\operatorname{dom} R$ has an R-minimal element; and a Σ_1 characterization, namely there is a ranking function from R to Ord.

In the second-order context, we still have access to the Π^0_1 characterization.[1] To say that a class relation R is well-founded is to say that there is no descending ω-sequence in R. But such an object must be a set, as its members must be bounded in rank.

As such, well-foundedness is downward absolute to transitive models of second-order set theory; if $R \in \mathcal{X}$ is well-founded in V then $(M, \mathcal{X})$ thinks R is well-founded. However, well-foundedness for class relations is not upward absolute for transitive models of second-order set theory. The trouble is that class relations can have rank $>$ Ord so we cannot use the Δ^0_0 definition of the von Neumann ordinals to get a Σ^0_1 characterization for well-foundedness.(Of course, well-foundedness for *set* relations is upward absolute for transitive models.)

Similar to the definitions of the least transitive model of T, a T-realization for M, a T-realizable model, and the least T-realization for M we can define the *least β-model of T*, a *β-T-realization for M*, a *β-T-realizable* model, and the *least β-T-realization for M*. For instance, $\mathcal{X} \subseteq \mathcal{P}(M)$ is a *β-T-realization for M* if $(M, \mathcal{X}) \models T$ is a β-model.

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[1]To clarify, the superscript 0 and 1 refer to whether a formula includes quantification over classes, rather than the arithmetical/analytical hierarchies from descriptive set theory. The 0 is for first-order quantification; for instance, a Π^0_1 formula has a single unbounded universal set quantifier, with all other quantifiers bounded. The 1 is for second-order quantification; for instance, a Σ^1_1 formula is of the form $\exists X \varphi(X, P)$ where φ is Π^0_k for some k.

The convention in this paper when talking about formulae in the language of set theory is to use capital letters for classes and class variables and lowercase letters for sets and set variables. For example, the formula $\forall X \exists y\ y \in X$ gives the (false) proposition that every class has some set as a member.

Definition 4. Kelley-Morse set theory KM is the second-order set theory consisting of the following axioms:

- ZFC for the first-order part;
- Extensionality for classes;
- Class Replacement, asserting that the image of any set under a class function is a set;
- Global Choice, asserting that there is a bijection between V and Ord; and
- The Class Comprehension axiom schema, namely the axioms $\forall \bar{P} \exists A \forall x\ x \in A \Leftrightarrow \varphi(x, \bar{P})$, for every second-order formula $\varphi(x, \bar{P})$. That is, φ can include quantifier over sets or over classes.

Gödel–Bernays set theory GBC is axiomatized similarly, except that the Class Comprehension schema is weakened. Its axioms are ZFC for the first-order part, Extensionality for classes, Class Replacement, Global Choice, and what I will call the First-Order Comprehension schema. This is a subschema of the Class Comprehension schema where we restrict to only formulae which do not include class quantifiers.

Intermediate between GBC and KM is the theory ETR. The axioms of ETR consist of the axioms of GBC plus the Elementary Transfinite Recursion schema.[2] This schema asserts that if φ is a first-order formula (possibly with class parameters) and R is a well-founded class relation then there is a solution to the recursion of φ along R. Formally, let $\varphi(x, Y, A)$ be a first-order formula, possibly with a class parameter A and let R be a well-founded class relation. Denote by $<_R$ the transitive closure of R. The instance of ETR for φ and R asserts that there is a class $S \subseteq \operatorname{dom} R \times V$ which satisfies

$$S_r = \{x : \varphi(x, S \upharpoonright r, A)\}$$

for all $r \in \operatorname{dom} R$. Here, $S_r = \{x : (r, x) \in S\}$ denotes the r-th slice of S and

$$S \upharpoonright r = S \cap (\{r' \in \operatorname{dom} R : r' <_R r\} \times V)$$

is the partial solution below r.

One well-known class recursion is the Tarskian definition of truth. This recursion takes place over a (proper) class tree of height ω. Thus, ETR asserts the existence of a first-order truth predicate and hence proves Con(ZFC), as restricting the truth predicate to parameter-free formulae gives a consistent completion of ZFC.[3] This shows that ETR exceeds GBC in consistency strength.

For an upper bound, one can use Π^1_1-CA to show that there is no least failure of a potential solution to an elementary recursion and from that conclude that a solution must exist. Therefore, ETR is properly contained within Π^1_1-CA, the fragment of KM obtained by restricting Class Comprehension to Π^1_1 formulae. As KM proves the existence of Π^1_k-truth predicates, KM proves Con(Π^1_1-CA) and thus ETR is below KM in consistency strength.

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[2] Elsewhere in the literature, e.g. [Fuj12; GH17], ETR is used to refer specifically to the Elementary Transfinite Recursion schema, with GBC + ETR being what I call simply by ETR. For the purposes of this paper, it is less awkward to use ETR to refer to the entire theory.

[3] If the model is ω-nonstandard, meaning that its ω is ill-founded, things are a little more delicate. The model will still think that ZFC is consistent because what it thinks is ZFC, nonstandard instances of the axiom schemata and all, will be contained in what it thinks is the first-order theory of the model, which it necessarily thinks is complete and consistent.

A useful fact is that Elementary Transfinite Recursion is equivalent to having solutions to a certain class of recursions. See [Fuj12, Corollary 61] and [GH17, Theorem 8].

Theorem 5. *Over* GBC*, Elementary Transfinite Recursion is equivalent to the assertion that for every class A and every class well-order Γ there is a Γ-iterated first-order truth predicate relative to A.*

A Γ-iterated truth predicate is a class T of triples (a, φ, p) where $(a, \varphi, p) \in T$ intuitively means that $\varphi(p)$ is true at level $a \in \mathrm{dom}(\Gamma)$. Here, φ is in the language with $\in$ and a predicate symbol $\hat{T}$ for T. Formally, this is defined by a modified form of the Tarskian recursion, with an extra clause in the definition asserting that $(a, \ulcorner \hat{T}(b, \varphi, p) \urcorner, q) \in T$ if and only if $b <_\Gamma a$ and $(b, \varphi, p) \in T$. A truth predicate relative to A is for the language expanded with a predicate symbol $\hat{A}$ for A, with an extra clause in the definition asserting that $(a, \ulcorner \hat{A}(x) \urcorner, p) \in T$ if and only if $p \in A$.

It can be proven from ETR that any two would-be Γ-iterated truth predicates relative to A must be the same class: they must agree on the truth of atomic assertions, by the definition of an iterated truth predicate, and inductively must agree on the truth of more complicated assertions. I will use $\mathrm{Tr}_\Gamma(A)$ to refer to the unique Γ-iterated truth predicate relative to A. Iterated truth predicates not relative to any class will be denoted Tr_Γ. As a special case, $\mathrm{Tr}(A) = \mathrm{Tr}_1(A)$ is the ordinary truth predicate relative to A.

I include a brief proof sketch for theorem 5 below. For a full proof see the above cited literature.

Proof sketch of theorem 5. ($\Rightarrow$) Construct $\mathrm{Tr}_\Gamma(A)$ by means of a recursion of rank $\omega \cdot \Gamma$.

($\Leftarrow$) Given $\mathrm{Tr}_\Gamma(A)$ one can construct a solution to a recursion of φ (with parameter A) along a relation of rank Γ. □

As an immediate corollary we get the following.

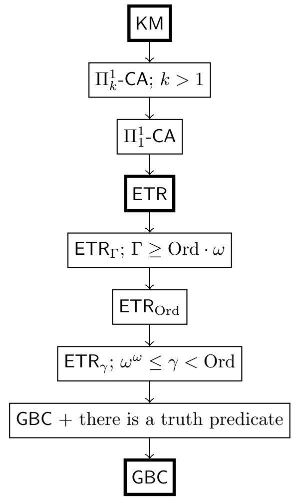
Corollary 6. *If $\omega^\omega \leq \Gamma$ then* GBC *plus the principle of Elementary Transfinite Recursion for relations of rank $\leq \Gamma$ is equivalent over* GBC *to the existence of Γ-iterated truth predicates relative to any class.*

Proof. The assumption on Γ implies that $\omega \cdot \Gamma < \Gamma + \Gamma$. Because ETR_Γ implies $\mathsf{ETR}_{\Gamma+\Gamma}$ (and $\mathsf{ETR}_{\Gamma \cdot n}$ for all standard n), this gives that we can do the necessary recursions of rank $\omega \cdot \Gamma$. □

For a class well-order Γ, let ETR_Γ denote the principle of Elementary Transfinite Recursion restricted to relations of rank $\leq \Gamma$. It must be clarified what this means. If $\Gamma \in \mathcal{X}$ then it makes sense to ask whether $(M, \mathcal{X})$ satisfies ETR_Γ—simply use Γ as a parameter. However, this approach flounders upon the questions of whether there is a least β-model of or a least transitive model of ETR_Γ. How should Γ be interpreted if we do not already have a model in mind?

For a more robust notion, let Γ be a name for a well-order. That is, Γ is defined by some formula which GBC proves defines a well-order. I will slightly abuse notation and use ETR_Γ in this context as well. It denotes GBC plus the principle of Elementary Transfinite Recursion restricted to relations of rank $\leq \Gamma$, where Γ is a name for a well-order. It then makes sense to ask, for instance, whether there is a least transitive model of ETR_Γ. One particular natural instance of this is $\mathsf{ETR}_{\mathrm{Ord}}$, asserting that recursions of rank $\leq \mathrm{Ord}$ have solutions.[4]

[4]To briefly attempt to justify the naturalness of $\mathsf{ETR}_{\mathrm{Ord}}$, Gitman, Hamkins, Holy, Schlict, and myself showed [GHHSW] that $\mathsf{ETR}_{\mathrm{Ord}}$ exactly captures the strength of the forcing theorem for class forcings.



Some second-order set theories from GBC to KM, separated by consistency strength.

2.1. **Models of second-order set theories.** In the interest of having a self-contained presentation, I will summarize some well-known facts about how we construct models of these second-order set theories.

Let $M \models \mathsf{ZFC}$. Say that $A \subseteq M$ is *strongly amenable (to M)* if $(M, A) \models \mathsf{ZFC}(A)$, where $\mathsf{ZFC}(A)$ is the expansion of ZFC obtained by allowing a predicate symbol for A in the axiom schemata of ZFC. If A is strongly amenable then A is *amenable*, meaning that for $A \cap x$ is a set in M for any $x \in M$. (Formally, there is $y \in M$ so that $M \models z \in y$ if and only if $(M, A) \models z \in x$ and $z \in A$.) Observe that every class in a model of GBC is strongly amenable to the first-order part.

The following folklore observation is key in constructing models of GBC.

Observation 7. *Let M be a model of* ZFC *and let G be a strongly amenable global well-order of M. Then* $\mathrm{Def}(M, G)$ *is a* GBC*-realization for M, where* $\mathrm{Def}(M, G)$ *is the collection of classes of M first-order definable from G and set parameters.*

If M has a definable (from parameters) global well-order then $\mathrm{Def}(M) = \mathrm{Def}(M, \emptyset)$ is a GBC-realization for M. Note that having a definable global well-order is equivalent to satisfying the axiom $\exists x\ V = \mathrm{HOD}(\{x\})$, where $\mathrm{HOD}(\{x\})$ denotes the (definable) class of all sets hereditarily definable from ordinal parameters and x. (Having a global well-order definable without parameters is equivalent to $V = \mathrm{HOD}$.) Therefore, any model of of $\mathsf{ZFC} + \exists x\ V = \mathrm{HOD}(x)$ is GBC-realizable.

Any countable model of ZFC has a GBC-realization, even if it does not have a definable global well-order. This can be seen by a class forcing argument. If C is a Cohen-generic subclass of Ord, then by density C codes a strongly amenable global well-order. If $M \models \mathsf{ZFC}$ is countable we can find such $C \in V$ as we only need to meet countably many dense definable classes of M. This gives us $(M, \mathrm{Def}(M, C)) \models \mathsf{GBC}$ in V. As a consequence, GBC is conservative over ZFC. That is, any first-order theorem of GBC is a theorem of ZFC.

This construction does not generalize to uncountable models of ZFC. Say that a model of set theory is *rather classless* if its only amenable classes are its classes definable from set parameters. Note that the height of a rather classless model must have uncountable cofinality. Any ω-sequence cofinal in the ordinals of a model will be amenable because its initial segments are finite. It cannot, however, be definable for danger of contradicting Replacement.

Definition 8. If M is a model of set theory then an extension $N \supseteq M$ is a *top extension* if M is a rank-initial segment of N. Intuitively, any new elements in N come above the "top" of M.

Note that top extensions are end extensions—where N is an *end extension* of M if M is a transitive subclass of N—but the converse does not in general hold. However, if N is an elementary end extension of $M \models \mathsf{ZFC}$ then it must be a top extension: by elementarity, $V_\alpha^M = V_\alpha^N$ cannot have any new elements.

Theorem 9 (Keisler [Kei74], Shelah [She78])**.** *Any countable model of* ZFC *has an elementary top extension to a rather classless model.*[5]

If we start with countable $M \models \mathsf{ZFC}$ with no definable global well-order, then any rather classless N which is an elementary top extension of M has no amenable global well-order and therefore has no GBC-realization.

For ETR and KM, we cannot hope for techniques as general as in the GBC case. A significant obstacle is that there are countable models of ZFC without a KM-realization. This is a consequence of the fact that ETR, and thus also KM, proves Con(ZFC). Consequently, a model of ZFC can fail to have an ETR-realization simply by having the wrong theory; no model of $\mathsf{ZFC} + \neg\operatorname{Con}(\mathsf{ZFC})$ can be ETR-realizable. We can also separate ETR and KM in a similar way; if $(M, \mathcal{X}) \models \mathsf{ETR} + \neg\operatorname{Con}(\mathsf{ETR})$ then M cannot be KM-realizable.

A natural question is to characterize which (countable) models of ZFC are KM-realizable. (Or which models are ETR-realizable.) It is obviously necessary that the model satisfy the first-order consequences of KM. But this is in fact insufficient. Indeed, there is no theory whose models are all KM-realizable.

Theorem 10 (Folklore)**.** *There is no first-order theory all of whose models (or even just whose countable models) are* KM*-realizable. That is, given any consistent T extending* ZFC *there is a countable model of T which has no* KM*-realization.*

The theorem also holds if KM is replaced by ETR or any other second-order set theory which proves the existence of a first-order truth predicate.

Proof. Suppose that M is a model of T and $\mathcal{X}$ is a KM-realization for M. (If no model of T has a KM-realization we are already done.) We can assume without loss that M and $\mathcal{X}$ are countable.

Let $S \in \mathcal{X}$ be the truth predicate for M. By reflection, there is a club of ordinals $\alpha \in \operatorname{Ord}^M$ so that $(V_\alpha^M, S \cap V_\alpha^M)$ thinks that $S \cap V_\alpha^M$ satisfies the recursive Tarskian definition of truth. If M is ω-standard, meaning that ω^M is well-founded, then S is truth for M. If M is ω-nonstandard, then S restricted to the standard formulae is truth for M. In either case, we get that $V_\alpha^M \subseteq M$ satisfies the full elementary diagram of M and hence is elementary in M. Moreover, these V_α^M form an elementary tower which unions up to M. Altogether, we have seen that the first-order part of any model of KM must be the union of an elementary tower from its V_αs.

Let $N = V_{\alpha_0}^M$, where α_0 is the least ordinal as in the previous paragraph. Then, there is no $\gamma \in \operatorname{Ord}^N$ so that V_γ^N is elementary in N. Otherwise, $(N, S \cap N)$ would satisfy that the truth predicate for V_γ^N is $S \cap N \cap V_\gamma^N = S \cap V_\gamma^N$. Then, as $V_\gamma^N = V_\gamma^M$, we get by elementarity that (M, S) satisfies that the truth predicate for V_γ^M is $S \cap V_\gamma^M$. Therefore, V_γ^M is elementary in M, contradicting the leastness of α_0.

This model N is the desired counterexample. We have $N \models T$ yet N cannot be KM-realizable, as it is not the union of an elementary tower from its V_αs. □

[5] Keisler showed this theorem under the assumption of ◇ and the assumption of ◇ was later eliminated by Shelah.

The simplest way to produce a model of KM comes from looking at inaccessible cardinals: if κ is inaccessible, then $(V_\kappa, V_{\kappa+1}) \models \mathsf{KM}$. Of course, once we have any transitive model of KM we get countable transitive models by Löwenheim–Skolem.

Indeed, $(V_\kappa, V_{\kappa+1})$ satisfies more than just KM. It can easily be checked that it satisfies the following theory KM^+.

Definition 11. Let KM^+ denote the theory consisting of the axioms of KM plus the Class Collection axiom schema. This schema asserts that if for every set there is a class satisfying some property, then there is a single class coding the "meta-class" consisting of a witnessing class for every set. Formally, instances of this schema take the form

$$\forall P \left[(\forall x \exists Y \ \varphi(x, Y, P)) \Rightarrow (\exists Z \forall x \ \varphi(x, (Z)_x, P) \right],$$

where $(Z)_x = \{y : (x, y) \in Z\}$.

Indeed, every model of KM^+ looks like the $(V_\kappa, V_{\kappa+1})$ of some appropriate model of set theory. Let $\mathsf{ZFC}^-_\mathsf{I}$ denote the theory whose axioms are those of ZFC^- (namely, the axioms of ZFC without Powerset but axiomatized with Collection instead of Replacement[6]) along with the assertion that there is a largest cardinal κ and that κ is inaccessible. To be clear since in this context not all sets have powersets, the assertion that κ is inaccessible is: κ is regular and 2^λ exists with $2^\lambda < \kappa$ for all $\lambda < \kappa$. In particular, in a model of $\mathsf{ZFC}^-_\mathsf{I}$ the only sets without a powerset are those of size κ. Marek and Mostowski [MM75] showed that KM^+ and $\mathsf{ZFC}^-_\mathsf{I}$ are bi-interpretable. If $M \models \mathsf{ZFC}^-_\mathsf{I}$ with largest cardinal $\kappa \in M$ then $(V^M_\kappa, V^M_{\kappa+1}) \models \mathsf{KM}^+$. For the other direction, looking at the definable meta-class of well-founded, extensional classes of a model of KM^+ and modding out by isomorphism gives a model of $\mathsf{ZFC}^-_\mathsf{I}$. This bi-interpretability result is useful when working with models of KM^+, as it allows one to instead work with models of a first-order set theory, allowing additional tools.

Given $(M, \mathcal{X}) \models \mathsf{KM}^+$ I will call the model of $\mathsf{ZFC}^-_\mathsf{I}$ obtained by $(M, \mathcal{X})$ in this manner the *unrolling of* $(M, \mathcal{X})$. The intuition is that the well-founded extensional classes in $\mathcal{X}$ are copies of the sets in the $\mathsf{ZFC}^-_\mathsf{I}$ model which have been rolled-up to lower their rank as much as possible.

When looking at the phenomenon of least models moving from KM to KM^+ is harmless.

Theorem 12 (Chuaqui [Chu80, theorem 12.3])**.** *Let $(M, \mathcal{X}) \models \mathsf{KM}$. Then there is $\mathcal{Y} \subseteq \mathcal{X}$ so that $(M, \mathcal{Y}) \models \mathsf{KM}^+$.*

This $\mathcal{Y}$ is obtained by $\mathcal{X}$ by a second-order analogue of Gödel's construction of L within V.

3. A detour through admissible set theory

In order to obtain main theorem 1 I will make use of some concepts from the theory of admissible structures. Recall that Kripke–Platek set theory KP is the (first-order) set theory consisting of the axioms of Extensionality, Pairing, Union, and Foundation, along with the Separation and Collection schemata for Σ_0 formulae. Transitive models of KP are known as *admissible sets*. To each admissible set A can be associated the infinitary logic L_A. The formulae in L_A can be given by the following schema:

- All atomic formulae are in L_A;
- If φ is in L_A then so is $\neg\varphi$;
- If $\Phi \in A$ is a set of formulae in L_A then $\bigvee_{\varphi \in \Phi} \varphi$ and $\bigwedge_{\varphi \in \Phi} \varphi$ are in L_A; and

[6] In the absence of Powerset, Collection is stronger than Replacement [Zar96].

- If φ is in L_A then so are $\exists x \varphi$ and $\forall x \varphi$.

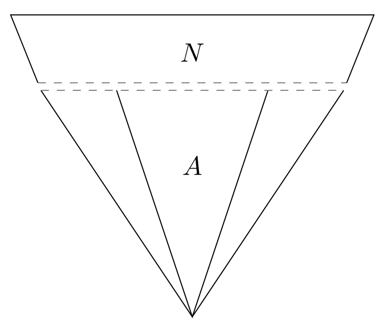
Note that $L_A \subseteq A$. Satisfaction for L_A theories is defined by the usual Tarskian recursion, extended transfinitely.

The following theorem, due to H. Friedman [Fri73], can be used to produce models N of set theory which are "ill-founded at an admissible set A", meaning that N is an end-extension of A and there is a descending ω-sequence coinitial in $\mathrm{Ord}^N \setminus \mathrm{Ord}^A$. Such models will have that $\mathrm{wfp}(N)$, the well-founded part of N, has the same height as and contains A. But in general it is not possible to insist that $\mathrm{wfp}(N)$ is exactly A. For example, suppose we want a model N of $\mathsf{ZFC} + 0^\sharp$ exists which is ill-founded at $L_{\omega_1^{\mathrm{CK}}}$, the least admissible set which has ω as an element. Because $0^\sharp$ is a subset of ω, it must be in $\mathrm{wfp}(N)$ but $0^\sharp$ is not in $L_{\omega_1^{\mathrm{CK}}}$.

Theorem 13 ([Fri73, theorem 2.2])**.** *Let A be a countable admissible set and let $T \subseteq A$ be an L_A theory which is Σ_1-definable in A and extends* KP*. If there is a model of T which contains A then there is an ill-founded model N of T so that*

- *N is an end extension of A;*
- $\mathrm{wfp}(N) \supseteq A$*; and*
- $\mathrm{Ord}^{\mathrm{wfp}(N)} = \mathrm{Ord}^A$*.*

The following picture illustrates the theorem. The dashed line represents that N is ill-founded at A.



I will use a strengthening of this theorem which applies to ill-founded models of KP as well as transitive models. The difficulty is that the elementary tools of admissible set theory only apply directly to transitive models of KP. To overcome this, Barwise introduced the notion of the admissible cover.[7] The admissible cover of a (possibly ill-founded) model A of set theory is an admissible structure with A as its urelements. We can work with this admissible structure to prove results about A.

Recall that KPU is KP formulated with urelements. I will write $\mathfrak{M} = (M; U; R_1, \ldots)$ for the transitive model of KPU with sets M, urelements U, and additional relations, functions, or constants $R_1, \ldots$. Let (A, E) be a possibly ill-founded model of KP. Say that such an admissible structure $\mathfrak{M} = (M; A; E, F)$ *covers* A if this additional function F maps urelements to their set of E-predecessors. That is, if $p \in A$ then $F(p) = \{q \in A : q \mathrel{E} p\} \in M$. (This function allows us to translate bounded quantification in the language of A to bounded quantification in the language of M. Quantifiers of the form $\forall p \mathrel{E} q$ can be replaced with $\forall p \in F(q)$.)

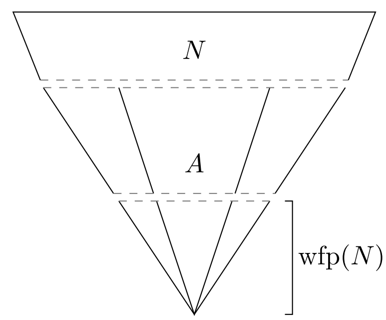
[7] Details and proofs of the properties of the admissible cover can be found in the appendix of [Bar75].

There is a smallest admissible structure which covers A, known as the *admissible cover* of A and denoted Cov_A. This structure enjoys many nice properties.

Proposition 14. *Let* $(A, E) \models \mathsf{KP}$ *and* $\mathfrak{C} = \mathrm{Cov}_A$.

- $\mathfrak{C}$ *is countable if* A *is countable.*
- *The pure sets of* $\mathfrak{C}$ *are isomorphic to the well-founded part of* A.
- *For any* $a \subseteq A$, *we have* $a \in \mathfrak{C}$ *if and only if there is* $x \in A$ *so that* $a = \{y \in A : A \models Y \, E \, x\}$. *As a consequence, if* $x \in A$ *then there is an* $L_{\mathfrak{C}}$ *sentence defining* x.
- *The infinitary diagram of* A, *considered as a set of* $L_{\mathfrak{C}}$*-sentences, is* Σ_1*-definable in* $\mathfrak{C}$.

We are now in a position to generalize Friedman's theorem to the ill-founded case. We want, when starting with a possibly ill-founded model A to produce the same picture as above. That is, given a theory T satisfying an appropriate consistency assumption, we want $N \supseteq A$ a model of T which is ill-founded at A.



Theorem 15. *Let* $(A, E^A) \models \mathsf{KP}$ *be countable and* $\mathfrak{C} = \mathrm{Cov}_A$. *Suppose that* T *is an* $L_{\mathfrak{C}}$ *theory which is* Σ_1*-definable over* $\mathfrak{C}$ *and extends* KP. *If there is a model of* T *which contains* A *then there is* $(N, E^N) \models T$ *so that:*

- $A \subsetneq N$;
- Ord^A *is a proper initial segment of* Ord^N;
- *There is an* ω*-sequence coinitial in* $\mathrm{Ord}^N \setminus \mathrm{Ord}^A$.

Proof Sketch. Friedman's original proof can be modified to give a proof this variant. Extend T, if necessary, to include the infinitary diagram of A. This extension is consistent as there is a model of T containing A. We want to construct a further extension T' so that, for countably many new constants c_n, the following conditions hold:

(1) Each $\varphi \in T'$ is consistent;
(2) For $\varphi \in L_{\mathfrak{C}}$, either $\varphi \in T'$ or "$\neg\varphi$" $\in T'$;
(3) For $\bigwedge \Phi \in L_{\mathfrak{C}}$ if $\Phi \subseteq T'$ then $\bigwedge \Phi \in T'$;
(4) For "$\forall x \varphi(x)$" $\in L_{\mathfrak{C}}$ if for each c_n we have $\varphi(c_n) \in T'$ then "$\forall x \varphi(x)$" $\in T'$;
(5) For each $a \in A$, "$a \in c_0$" $\in T'$; and
(6) If "$a \in c_n$" $\in T'$ for each $a \in A$ then there is $m > n$ so that "$c_m \in c_n$" $\in T'$ and "$a \in c_m$" $\in T'$ for each $a \in A$.

Conditions (1–3) ensure there is a model of T'. Condition (5) forces any model of T' to contain new ordinals. Conditions (4–6) force that the model of T' is ill-founded above A. To see this note, that if $\beta < \gamma$ are ordinals above Ord^A but below $\inf\{\mathrm{rank}(c_n) : c_n \text{ above } A\}$ then for every c_n "$\mathrm{rank}(c_n) \geq \beta \Rightarrow \mathrm{rank}(c_n) > \gamma$" $\in T'$ so by condition (4) $\beta > \gamma$, a contradiction.

T' is constructed in ω many stages. We continually add new formulae to ensure properties (1–6) hold. Fix an enumeration $\langle\varphi_n\rangle$ of the $L_{\mathfrak{C}}$ sentences so that c_n first appears after φ_n and before the first appearance of c_{n+1}.

- Define $T'_0 = T \cup \{a \in c_0 : a \in A\}$. This theory is consistent by Barwise compactness. This ensures property (5). Set $m_0 = 1$ and $m_{-1} = 0$.
- For $n \geq 0$, define T'_{3n+1} to be $T'_{3n} \cup \{\psi\}$, where ψ is chosen from φ_n and $\neg\varphi_n$ so as to be consistent with T'_{3n}. This step will ensure properties (1) and (2).
- For $n \geq 0$, define T'_{3n+2} as follows, according to which of three cases we fall into.
  - If φ_n is of the form $\bigwedge\Phi$ and $\neg\varphi_n$ is in T'_{3n+1}, then take $T'_{3n+2} = T'_{3n+1} \cup \{\neg\varphi\}$ where $\varphi \in \Phi$ and this is consistent. This step ensures property (3).
  - If φ_n is of the form $\forall x\psi(x)$ and $\neg\varphi_n$ is in T'_{3n+1}, then take $T'_{3n+2} = T'_{3n+1} \cup \{\neg\psi(c_m)\}$, where m is the index of the least unusued c_m. This step ensures property (4).
  - Otherwise, just take $T'_{3n+2} = T'_{3n+1}$.
- For $n \geq 0$, define T'_{3n+3} as follows, according to which of two cases we fall into.
  - If there is $a \in A$ so that $T'_{3n+2} \cup \{a \notin c_{m_n}\}$ is consistent, take this to be T'_{3n+3}. Set $m_{n+1} = m_n$.
  - Otherwise, the theory $T'_{3n+2} \cup \{a \in c_{m_n} : a \in M\}$ is consistent. By Barwise compactness, so is the the theory $T'_{3n+2} \cup \{a \in c_{m_n} : a \in A\} \cup \{c_{m_n} \in c_{m_{n-1}}\}$. Take this to be T'_{3n+3} and set m_{n+1} to be the index of the least unused c_m.
- Set $T' = \bigcup_n T'_n$.

By the construction for T'_{3n+3}, for every $n \geq -1$ we have that $c_{m_{n+1}} \in c_{m_n}$ and $a \in c_{m_n}$, for all $a \in A$ are in T'. This gives property (6). □

4. GBC

Let us begin with a classical result.

Theorem 16 (Shepherdson [She53]). *If there is a transitive model of* GBC *then there is a least transitive model of* GBC*.*

Proof. Let L_α be the least transitive model of ZFC. Then $(L_\alpha, \mathrm{Def}(L_\alpha)) = (L_\alpha, L_{\alpha+1})$ is the least transitive model of GBC. That it satisfies Global Choice is because the L-order is definable. If $(M, \mathcal{X})$ is a model of GBC whose height is α, then $L_{\alpha+1} \subseteq \mathcal{X}$ because L is a definable class. If the height of M is greater than α, then $L_{\alpha+1} \subseteq M$. □

Essentially the same argument gives the following.

Corollary 17. *If there is a β-model of* GBC *then there is a least β-model of* GBC*.*

The key to proving this is the following observation.

Observation 18.

(1) *Let $M \models$ ZFC. If $(M, \mathcal{X})$ is a β-model and $\mathcal{Y} \subseteq \mathcal{X}$ then $(M, \mathcal{Y})$ is a β-model.*
(2) *Moreover, if $(M, \mathcal{X})$ is a β-model and $(N, \mathcal{Y})$ is such that $\mathrm{Ord}^N = \mathrm{Ord}^M$, $N \subseteq M$, and $\mathcal{Y} \subseteq \mathcal{X}$, then $(N, \mathcal{Y})$ is a β-model.*

Proof. (1) Suppose $R \in \mathcal{Y}$ is ill-founded. Because $(M, \mathcal{X})$ is a β-model, $(M\,\mathcal{X})$ thinks R is ill-founded. This is witnessed by a countable descending sequence d in R. But then $d \in M$ so $(M, \mathcal{Y})$ thinks R is ill-founded.

(2) Suppose $R \in \mathcal{Y}$ is ill-founded. Again, $(M, \mathcal{X})$ thinks that R is ill-founded, so there is $d \in M$ witnessing this. Let $\theta \in \mathrm{Ord}^M$ be so that $d \in V_\theta^M$. Set $r = R \cap V_\theta^N \in N$. Then $r \in M$ because $N \in \mathcal{Y} \subseteq \mathcal{X}$. Observe that r is ill-founded, as M sees that d witnesses the ill-foundedness of r. But r is a set in N which is a transitive model of ZFC. So N must think r is ill-founded and hence $(N, \mathcal{Y})$ thinks R is ill-founded. □

Proof of corollary 17. Consider $(L_\alpha, \mathrm{Def}(L_\alpha))$, where α is least such that there is a β-model of GBC of height α. Let $(M, \mathcal{X}) \models \mathsf{GBC}$ be a β-model with $\mathrm{Ord}^M = \alpha$. Then $L_\alpha \supseteq M$ and $\mathrm{Def}(L_\alpha) \subseteq \mathcal{X}$. By the observation, $(L_\alpha, \mathrm{Def}(L_\alpha))$ is a β-model. It must be contained in every β-model of GBC. □

More interesting is the question of when there is a least GBC-realization for a model of ZFC.

Theorem 19. *Let M be a countable model of* ZFC*. Then M has a least* GBC*-realization if and only if M has a definable global well-order.*

Proof. ($\Leftarrow$) Any GBC-realization for M must contain $\mathrm{Def}(M)$. If M has a definable global well-order then $\mathrm{Def}(M)$ is itself a GBC-realization, hence it is the least GBC-realization for M.

($\Rightarrow$) Suppose towards a contradiction that $\mathcal{X}$ is the least GBC-realization for M. By leastness and as M has no definable global well-order, it must be that $\mathcal{X} = \mathrm{Def}(M, G)$, where G is some global well-order of M. Consider the forcing to add a Cohen subclass to Ord with set-sized conditions. Note that this forcing does not add any new sets as it is κ-closed for every κ. Let H be $(M, \mathcal{X})$-generic for this forcing. That is, H meets every dense class in $\mathcal{X}$. Given any set a, by density a is coded into H. This yields a global well-order definable from H: say that $a <_H b$ if the index where a is first coded into H is less than the index where b is first coded into H. Therefore, $\mathrm{Def}(M, H)$ is a GBC-realization for M.

I claim that $G \notin \mathrm{Def}(M, H)$. Otherwise, there is a first-order formula φ, possibly with set parameters and H as a class parameter but where G does not appear as a parameter, so that $(M, \mathrm{Def}(M, G, H)) \models \forall x\ x \in G \Leftrightarrow \varphi(x, H)$. There is some $p \in H$ forcing this statement; that is, there is $p \in H$ so that $(M, \mathrm{Def}(M, G)) \models$ "$p \Vdash \forall x\ x \in \check{G} \Leftrightarrow \varphi(x, \dot{H})$". Thus, $(M, \mathrm{Def}(M, G)) \models$ "$\forall x\ x \in G \Leftrightarrow p \Vdash \varphi(\check{x}, \dot{H})$" so for all $x \in M$, we have $x \in G$ if and only if $(M, \mathrm{Def}(M, G)) \models p \Vdash \varphi(\check{x}, \dot{H})$. But this formula doesn't depend upon G, so this happens if and only if $M \models p \Vdash \varphi(\check{x}, \dot{H})$. Therefore $G \in \mathrm{Def}(M)$, contradicting that M has no definable global well-order.

As $G \notin \mathrm{Def}(M, H)$, it cannot be that $\mathcal{X} = \mathrm{Def}(M, G) \subseteq \mathrm{Def}(M, H)$, contradicting the leastness of $\mathcal{X}$. □

As remarked in Section 2, a model of ZFC having a definable global well-order is equivalent to that model satisfying the first-order sentence $\exists x\ V = \mathrm{HOD}(\{x\})$. Thus, whether a countable model of set theory has a least GBC-realization is recognizable from the theory of the model.

Corollary 20. *A countable model of* ZFC *has a least* GBC*-realization if and only if it satisfies $\exists x\ V = \mathrm{HOD}(\{x\})$.* □

The construction used in the proof of theorem 19 does not show show that any countable $M \not\models \exists x\ V = \mathrm{HOD}(\{x\})$ has no minimal GBC-realization. The reason is that the $\mathrm{Def}(M, G)$ and $\mathrm{Def}(M, H)$ from the proof are incomparable. We have already seen one direction, that $\mathrm{Def}(M, G) \not\subseteq \mathrm{Def}(M, H)$. The other direction, that $\mathrm{Def}(M, H) \not\subseteq \mathrm{Def}(M, G)$, is because by genericity $H \notin \mathrm{Def}(M, G)$. We get that $\mathrm{Def}(M, G)$ is not least, but left open is the possibility that it could be minimal.

Question 21. *Is there $M \models \mathsf{ZFC} + \forall x\ V \neq \mathrm{HOD}(\{x\})$ with a minimal* GBC*-realization $\mathcal{X}$ for M? Can this happen for countable M?*

Answering this question in general seems difficult, but something can be said for special cases. First is a negative result that minimal GBC-realizations cannot come from Cohen forcing.

Proposition 22. *Consider $M \models \mathsf{ZFC}$ and suppose $C \subseteq \mathrm{Ord}^M$ is Cohen-generic. Then, $\mathrm{Def}(M, C)$ is not a minimal* GBC*-realization.*

Proof. Define C' as $\alpha \in C'$ if and only if $\alpha \cdot 2 + 1 \in C$. If we think of C as an Ord-length binary sequence, C' is what appears on the odd coordinates of C. It is a standard result about Cohen forcing that C' is Cohen-generic but $C \notin \mathrm{Def}(M, C')$. Thus, $\mathrm{Def}(M, C') \subsetneq \mathrm{Def}(M, C)$ and so $\mathrm{Def}(M, C)$ is not minimal. □

If we want a model of ZFC with minimal but not least GBC-realizations, we must start with a model without a definable global well-order. We want to add a strongly amenable global well-order to this model which is minimal in a certain sense. A natural place to look here is at generalizations of Sacks forcing. As is well-known, Sacks reals are minimal over the ground model. Can we generalize Sacks forcing to produce a minimal generic class of ordinals which codes a global well-order?

Generalizing Sacks forcing to add generic classes of ordinals has been considered before. Hamkins, Linetsky, and Reitz [HLR13] consider so-called "perfect generics", an adaptation of a technique of Kossak and Schmerl [KS06] from models of arithmetic. Kossak and Schmerl's technique is in turn an adaptation from Sacks forcing over ω. They produce minimal generics for countable models M of arithmetic, i.e. inductive $G \subseteq M$ so that for $A \in \mathrm{Def}(M, G)$ either $A \in \mathrm{Def}(M)$ or $G \in \mathrm{Def}(M, A)$.

I do not see a way to use perfect generics to produce minimal but not least GBC-realizations. Nevertheless, a related result can be achieved.

Theorem 23. *Let $(M, \mathcal{X}) \models \mathsf{GBC}$ be countable. Then there is $\mathcal{Y} \supsetneq \mathcal{X}$ a* GBC*-realization which is minimal above $\mathcal{X}$: if $\mathcal{Z} \supsetneq \mathcal{X}$ is a* GBC*-realization, then $\mathcal{Y} \subseteq \mathcal{Z}$.*

This theorem shows that the poset consisting of countable GBC-realizations of a fixed countable $M \models \mathsf{ZFC}$ is not dense in a strong sense. For any $\mathcal{X}$ in this poset there is $\mathcal{Y} \supseteq \mathcal{X}$ with no elements of the poset in-between them. On the other hand, proposition 22 gives $\mathcal{Z} \supseteq \mathcal{X}$ so that the interval $(\mathcal{X}, \mathcal{Z})$ of intermediate GBC-realizations contains a dense linear order.

Before proving this theorem, it will be necessary to set up some machinery.

Work with a fixed countable $\mathfrak{M} = (M, \mathcal{X}) \models \mathsf{GBC}$. Let $\mathbb{B}$ denote the full Ord-length binary tree. For a perfect tree $\mathbb{P} \subseteq \mathbb{B}$, say that $\mathbb{Q} \subseteq \mathbb{P}$ is *n-deciding for* $\mathbb{P}$ if for every Σ_n formula φ in the forcing language for M, there is an ordinal α so that if $p \in \mathbb{Q}$ has length greater than α then p decides φ. It is not hard to see that if $\mathbb{P} \in \mathcal{X}$ then we can find $\mathbb{Q} \in \mathcal{X}$ which is n-deciding for $\mathbb{P}$. Moreover, we can find such $\mathbb{Q}$ which does not split below a fixed $\alpha \in \mathrm{Ord}$.

Definition 24. A function $G : \mathrm{Ord} \to 2$ is called a *perfect generic* if there is a sequence $\mathbb{B} = \mathbb{P}_0 \supseteq \mathbb{P}_1 \supseteq \cdots \supseteq \mathbb{P}_n \supseteq \cdots$ of perfect trees from $\mathcal{X}$ so that $G = \bigcup(\bigcap_n \mathbb{P}_n)$ and $\mathbb{P}_{n+1}$ is n-deciding for $\mathbb{P}_n$ for all n.

A subtle point is that a perfect generic need not be fully generic over any of the $\mathbb{P}_n$. So the model generated by a perfect generic G, namely $\mathfrak{M}[G] = (M, \{X \subseteq M : X \in \mathrm{Def}(M, G, A)$ for some $A \in \mathcal{X}\})$, need not be a class forcing extension of $\mathfrak{M}$. Nevertheless, $\mathfrak{M}[G]$ will still satisfy GBC. Extensionality, Global Choice, and First-Order Comprehension are immediate. To see $\mathfrak{M}[G]$ satisfies Replacement, take a class function F. Then F is definable by a Σ_k-formula for some k with parametres G and $A \in \mathcal{X}$. But then the behavior of F is forced by conditions in $\mathbb{P}_{k+1} \supseteq G$.

Because perfect generics need not be fully generic, we need to refine some standard facts about class forcing. I summarize them here. First, the forcing relation (for pretame forcings, such as the posets we will use) for formulae of bounded complexity is definable. That is, the relation $p \Vdash_{\mathbb{P}} \varphi$, confined to φ which are Σ_k is Σ_j-definable from $\mathbb{P}$ for some $j > k$. (It does not matter for our purposes what this j is, just that it is finite.) Second, we need to refine the usual "forcing equals truth" lemma: for φ which are Σ_k there is $j > k$ so that for any $G \subseteq \mathbb{P}$ which is Σ_j-generic over $\mathfrak{M}$ then $\mathfrak{M}[G] \models \varphi$ if and only if there is $p \in G$ so that $p \Vdash_{\mathbb{P}} \varphi$. This can be proved via a standard inductive argument, attending to the complexity of the definitions of the dense sets involved. In short, the standard facts for forcing apply for partial generics, except that we must be careful to confine ourselves to things which are sufficiently simple.

The following lemmata will be used to prove the theorem. They are set theoretic analogues of results from section 6.5 of [KS06].

Lemma 25. *Let $\varphi(x)$ be a formula in the forcing language and $\mathbb{P} \in \mathcal{X}$ be a perfect subtree of $\mathbb{B}$. There is $\mathbb{Q} \subseteq \mathbb{P}$ in $\mathcal{X}$ so that one of the two cases holds:*

(1) *There is an ordinal α so that for all ordinals ξ we have that all $p \in \mathbb{Q}$ of length greater than α decide $\varphi(\check{\xi})$ (in $\mathbb{P}$) the same.*
(2) *For every ordinal α there is $\beta > \alpha$ so that if $p, q \in \mathbb{Q}$ both have length β and $p \upharpoonright \alpha = q \upharpoonright \alpha$ then there is an ordinal ξ so that p and q decide $\varphi(\xi)$ differently (in $\mathbb{P}$).*

Proof. Fix k so that φ is a Σ_k formula. Take $\mathbb{P}' \subseteq \mathbb{P}$ a k-deciding subtree for $\mathbb{P}$. We may assume that there is a function $f : \mathbb{B} \to \mathbb{P}'$ which embeds the full binary tree onto the splitting nodes of $\mathbb{P}'$ and that $f(s)$ decides $\varphi(\operatorname{len} s)$. There are two cases. The first is that there is some $s \in \mathbb{B}$ so that for every $t, t' >_{\mathbb{B}} s$ if $\operatorname{len} t = \operatorname{len} t'$ then $f(t)$ and $f(t')$ decide $\varphi(\xi)$ the same for all ordinals ξ. In this case, set $Q = \mathbb{P}' \upharpoonright f(s)$ and get the first conclusion in the lemma.

The second case is that this does not happen for any $s \in \mathbb{B}$. In this case, we can inductively define a $g : \mathbb{B} \to \mathbb{P}'$ as follows:

- Set $g(0) = f(0)$.
- Set $g(s^\frown 0) = p_0$ and $g(s^\frown 1) = p_1$, where p_0, p_1 are least (according to a fixed global well-order) so that $\operatorname{len} p_0 = \operatorname{len} p_1$ and there is an ordinal ξ so that p_0 and p_1 decide $\varphi(\xi)$ differently. Such p_0 and p_1 always exist, as otherwise we would be in the previous case.
- At limit stages take unions.

Set $\mathbb{Q} = \{p \in \mathbb{P}' : \exists s \in \mathbb{B}\ p \leq_{\mathbb{P}'} g(s)\}$. This yields the second conclusion in the lemma. □

Observe that we used global choice in an essential manner here. There are possibly many choices for p_0 and p_1 in the successor stage of the construction of g. In order to guarantee that $g \in \mathcal{X}$ and hence that $\mathbb{Q} \in \mathcal{X}$, we need to uniquely specify a choice.

This puts us in a position to prove a minimality lemma for perfect generics.

Lemma 26. *Let $(M, \mathcal{X}) \models \mathsf{GBC}$ be countable. Then there is a perfect generic $G \notin \mathcal{X}$ so that for any class A of ordinals definable from G (and possibly parameters from $\mathcal{X}$), either G is definable from A and parameters from $\mathcal{X}$ or else $A \in \mathcal{X}$.*

Proof. Fix a cofinal sequence $\langle \alpha_n \rangle$ of ordinals and an enumeration $\langle \varphi_n(x) \rangle$ of formulae in the forcing language. We construct a descending sequence of perfect trees

$$\mathbb{B} = \mathbb{Q}_0 \supseteq \mathbb{P}_0 \supseteq \mathbb{Q}_1 \supseteq \mathbb{P}_1 \supseteq \cdots \supseteq \mathbb{Q}_n \supseteq \mathbb{P}_n \supseteq \cdots$$

so that $\mathbb{P}_n$ is an n-deciding subtree of $\mathbb{Q}_n$ which does not split below α_n and $\mathbb{Q}_{n+1} \subseteq \mathbb{P}_n$ is as in the previous lemma for φ_n. Our perfect generic is $G = \bigcup(\bigcap_n \mathbb{P}_n)$.

Consider a class of ordinals A defined from G and parameters in $\mathcal{X}$, where $G \Vdash x \in A \Leftrightarrow \varphi_n(x)$. Consider $\mathbb{Q}_{n+1} \subseteq \mathbb{P}_n$. If the first case from the previous lemma holds, then $A \in \mathcal{X}$ because $\xi \in A$ if and only if for every $p \in \mathbb{Q}_{n+1}$ the length of p being sufficiently long implies that $p \Vdash_{\mathbb{P}_n} \varphi_n(\xi)$. If the second case of the previous lemma holds, then we can define G from A. In this case, $p \in \bigcap_n \mathbb{P}_n$ if and only if for every ordinal α there is $q >_{\mathbb{Q}_{n+1}} p$ of length greater than α so that $q \Vdash_{\mathbb{P}_n} \varphi_n(\xi) \Leftrightarrow \xi \in A$ for all ordinals ξ. From a definition of $\bigcap_n \mathbb{P}_n$ can easily be produced a definition for G. □

Proof of theorem 23. We start with countable $(M, \mathcal{X}) \models \mathsf{GBC}$. Let G be the perfect generic as in the minimality lemma. Set $\mathcal{Y}$ to be the classes definable from G and parameters in $\mathcal{X}$. Then, $\mathcal{Y}$ is a GBC-realization for M. Suppose that $\mathcal{Z}$ is a GBC-realization so that $\mathcal{X} \subseteq \mathcal{Z} \subseteq \mathcal{Y}$. Consider arbitrary $A \in \mathcal{Z}$. We may assume that A is a class of ordinals by mapping A, via a fixed bijection $V \to \mathrm{Ord}$ from $\mathcal{X}$, into the ordinals. By the minimality lemma, either G is definable from A or else $A \in \mathcal{X}$. Therefore, if $\mathcal{Z} \neq \mathcal{X}$ then G is definable from an element of $\mathcal{Z}$ (with parameters from $\mathcal{X}$) and hence $\mathcal{Y} = \mathcal{Z}$. □

As remarked earlier, Global Choice was used essentially in the proof of lemma 25. Proving this lemma without Global Choice would yield a construction for minimal but not least GBC-realizations. Namely, start with a countable $M \models \mathsf{ZFC}$ with no definable global well-order. Let $\mathcal{X} = \mathrm{Def}(M)$. Then $(M, \mathcal{X})$ is a model of GBC minus Global Choice. Applying the theorem to $(M, \mathcal{X})$ would yield a GBC-realization $\mathcal{Y}$ for M which is minimal above $\mathcal{X}$. But since any GBC-realization must contain $\mathcal{X}$, this would give that $\mathcal{Y}$ is a minimal GBC-realization for M.

Thus, the problem of constructing a minimal but not least GBC-realization can be reduced down to the problem of proving the minimality lemma for perfect generics without using choice. A similar question can be asked for ordinary Sacks forcing.

Question 27. *Is choice needed to prove the minimality lemma for Sacks forcing? That is, is it consistent that there are $M \models \mathsf{ZF} + \neg AC$, $s \subseteq \omega^M$ Sacks-generic over M, and $A \in M[s]$ so that $M \subsetneq M[A] \subsetneq M[s]$?*

5. ETR and its fragments

The main result of this section is that there is a smallest β-model of ETR. First, a warm-up.

Theorem 28. *If $M \models \mathsf{ZFC} + \exists x\ V = \mathrm{HOD}(\{x\})$ is β-ETR-realizable, then M has a least β-ETR-realization.*

Proof. Suppose $(M, \mathcal{Y}) \models \mathsf{ETR}$ is a β-model. Externally to $(M, \mathcal{Y})$ we will construct a ETR-realization $\mathcal{X} \subseteq \mathcal{Y}$ for M and then see that the construction for $\mathcal{X}$ gives the same ETR-realization no matter which $\mathcal{Y}$ we started with.

Let $\mathcal{X}_0 = \mathrm{Def}(M)$. Because M has a definable global well-order, $(M, \mathcal{X}_0) \models \mathsf{GBC}$. Now, given $\mathcal{X}_n$, define $\mathcal{X}_{n+1}$ to consist of all classes of M definable from elements of $\{\mathrm{Tr}_\Gamma(A) : A, \Gamma \in \mathcal{X}_n$ and Γ is a class well-order$\}$. Recall that $\mathrm{Tr}_\Gamma(A)$ denotes the Γ-iterated truth predicate relative to the class A, where Γ is a class well-order. A few remarks are in order to clarify why this definition is well-formed. First, note that $\mathcal{X}_0 \subseteq \mathcal{Y}$ and thus $(M, \mathcal{X}_0)$ is a β-model. Second, if $\mathcal{X}_n \subseteq \mathcal{Y}$ then $\mathcal{X}_{n+1} \subseteq \mathcal{Y}$. This is because $\mathcal{Y}$ must have iterated truth predicates relative to any of its classes and because $\mathcal{Y}$ and $\mathcal{X}_n$ agree on what class relations are well-founded, both being β-models. Inductively, this yields that each $\mathcal{X}_n \subseteq \mathcal{Y}$ is a β-model. Because each of these is a β-model, they agree with V on whether Γ is a well-order. Finally, because M is transitive it has unique iterated truth predicates relative to a given parameter.

Set $\mathcal{X} = \bigcup_n \mathcal{X}_n$. Easily, $\mathcal{X} \subseteq \mathcal{Y}$. Because $\mathcal{X}$ is an increasing union of GBC-realizations, $\mathcal{X}$ itself is a GBC-realization for M. To see that $(M, \mathcal{X}) \models \mathsf{ETR}$, note that if $A, \Gamma \in \mathcal{X}$ then $A, \Gamma \in \mathcal{X}_n$ for some n and thus $\mathrm{Tr}_\Gamma(A) \in \mathcal{X}_{n+1} \subseteq \mathcal{X}$.

It remains only to see that we get the same $\mathcal{X}$ regardless of our choice of $\mathcal{Y}$. But this is immediate, since $\mathcal{Y}$ was not actually used to define the $\mathcal{X}_n$. We only used that $\mathcal{X}_n \subseteq \mathcal{Y}$ to conclude that $(M, \mathcal{X}_n)$ is a β-model, and this is true for any $\mathcal{Y}$ a β-ETR-realization for M. □

Theorem 29 (Main Theorem 2)**.** *There is a least β-model of ETR, if there is any β-model of ETR.*

Proof. First, let us see that if M is β-ETR-realizable then so is L^M. This is of independent interest, so I will separate it out as its own lemma.

Lemma 29.1. *If M is β-ETR-realizable then so is L^M.*

Before proving this, let us see why merely restricting the classes to those which are subclasses of L will not work. Suppose $(M, \mathcal{Y}) \models \mathsf{ETR}$ is transitive with $0^\sharp \in M$. Set $\mathcal{X} = \mathcal{Y} \cap \mathcal{P}(L^M)$. Then, $(L^M, \mathcal{X})$ fails to satisfy GBC, let alone ETR. Consider $A = 0^\sharp \cup (\mathrm{Ord}^M \setminus \omega) \in \mathcal{X}$. But $A \cap \omega \notin L^M$, so $(L^M, \mathcal{X})$ fails to satisfy Replacement.

Proof of lemma 29.1. Let $(M, \mathcal{Y}) \models \mathsf{ETR}$ be a β-model and let $\kappa = \mathrm{Ord}^M$. We want to find $\mathcal{X} \subseteq \mathcal{P}(L_\kappa) \cap \mathcal{Y}$ so that $(L_\kappa, \mathcal{X}) \models \mathsf{ETR}$. Then, by observation 18, $(L_\kappa, \mathcal{X})$ will be a β-model. This can be done by an argument analogous to the proof of theorem 28.

First, observe that $\mathcal{Y}$ contains iterated truth predicates for inner models. If $W \subseteq M$ is an inner model in $\mathcal{Y}$ then $\mathrm{Tr}^W_\Gamma(A)$, the Γ-iterated truth predicate for W relative to the parameter A, is also in $\mathcal{Y}$. This is because $\mathrm{Tr}^W_\Gamma(A)$ can be constructed via a transfinite recursion of rank $\omega \cdot \Gamma$.

Let $\mathcal{X}_0 = \mathrm{Def}(L_\kappa)$. Then $(L_\kappa, \mathcal{X}_0) \models \mathsf{GBC}$ is a β-model. Given $\mathcal{X}_n$ a β-GBC-realization for L_κ define $\mathcal{X}_{n+1}$ to consist of all classes of L_κ definable from $\mathrm{Tr}^{L_\kappa}_\Gamma(A)$ for $A, \Gamma \in \mathcal{X}_n$. Finally, set $\mathcal{X} = \bigcup_n \mathcal{X}_n$. Similar to before, we get that $(L_\kappa, \mathcal{X}) \models \mathsf{ETR}$: given $\Gamma, A \in \mathcal{X}$ there is n so that $\Gamma, A \in \mathcal{X}_n$ so $\mathrm{Tr}_\Gamma(A) \in \mathcal{X}_{n+1} \subseteq \mathcal{X}$.

There is, however, one new issue which must be addressed. In the proof of theorem 28 it was immediate that each $\mathcal{X}_n$ was a GBC-realization. The potential sticking point is Replacement. But because $\mathcal{X}_n \subseteq \mathcal{Y}$ had the same first-order part, Replacement for $\mathcal{Y}$ gave Replacement for $\mathcal{X}_n$. This is not in general true if the first-order part for $\mathcal{X}_n$ is smaller than the first-order part for $\mathcal{Y}$.

Here, however, we do get that $(L_\kappa, \mathcal{X}_{n+1})$ satisfies Replacement. To see this, fix $a \in L^M$ and a class function $F \in \mathcal{X}_{n+1}$. Because Γ really is well-founded, externally to M we can find an ordinal γ so that $\mathrm{otp}(\Gamma) = \gamma$. We can inductively show that $\mathcal{X}_{n+1} = \mathcal{P}(L_\kappa) \cap L_\xi$ where ξ is the supremum of the order-types of well-orders in $\mathcal{X}_n$. Note that $L_\xi \models \kappa$ is regular, as otherwise $\mathcal{Y}$ would contain a cofinal function $\alpha \to \mathrm{Ord}^M$ for some $\alpha \in \mathrm{Ord}^M$, which is impossible. Therefore, $L_\xi \models F(a) \in L_\kappa$. So $(L_\kappa, \mathcal{X}_{n+1})$ satisfies Replacement. □

As a consequence, the least β-model of ETR must have first-order part of the form L_α for some α.

Let L_α be so that α is the least ordinal with L_α being β-ETR-realizable. By theorem 28, let $\mathcal{X}$ be the least β-ETR-realization for L_α. Then, $(L_\alpha, \mathcal{X})$ is the desired least β-model of ETR. If $(M, \mathcal{Y}) \models \mathsf{ETR}$ is a β-model with $\mathrm{Ord}^M = \alpha$, then theorem 28 yields that $\mathcal{X} \subseteq \mathcal{Y}$. If $(M, \mathcal{Y}) \models \mathsf{ETR}$ with $\mathrm{Ord}^M > \alpha$ then $L_\alpha \in M$ and thus M can construct $\mathcal{X}$ by ordinary transfinite recursion on sets. □

Essentially the same argument gives least β-models for ETR_Γ.

Lemma 30. *If $M \models \mathsf{ZFC} + \exists x\ V = \mathrm{HOD}(\{x\})$ is β-ETR_Γ-realizable for $\Gamma \in \mathcal{P}(M)$ with $\Gamma \geq \omega^\omega$ then M has a least β-ETR_Γ-realization.*

The purpose of requiring $\Gamma \geq \omega^\omega$ is that this ensures ETR_Γ is equivalent to the existence of Γ-iterated truth predicates relative to any class. See corollary 6. The same applies to later theorems, but I will suppress making this comment every time.

Proof of lemma 30. This is done almost the same as the proof for theorem 28. Namely, set $\mathcal{X}_0 = \mathrm{Def}(M, \Gamma)$ and set $\mathcal{X}_{n+1}$ to consist of all clases of M definable from $\mathrm{Tr}_\Gamma(A)$ for $A \in \mathcal{X}_n$. Finally, set $\mathcal{X} = \bigcup_n \mathcal{X}_n$. These are well-defined because Γ is a well-order. Then, similar to before, $(M, \mathcal{X})$ is easily seen to be contained in any β-model $(M, \mathcal{Y}) \models \mathsf{ETR}_\Gamma$ and is itself a β-model of ETR_Γ. □

Theorem 31. *Let Γ be a name for a class well-order $\geq \omega^\omega$. That is, Γ is defined by some formula so that any model of GBC thinks the class defined by that formula is well-ordered. (For instance, Γ might be Ord.) If there is a β-model of ETR_Γ then there is a least β-model of ETR_Γ.*

Proof. This follows from lemma 30 plus the fact that if M is β-ETR_Γ-realizable then so is L^M. This fact can be proved similar to the argument for lemma 29.1, except that we only need iterated truth predicates of length Γ, rather than of arbitrary length. That is, start with $(M, \mathcal{Y}) \models \mathsf{ETR}_\Gamma$ a β-model. Set $\mathcal{X}_0 = \mathrm{Def}(L^M)$ and inductively set $\mathcal{X}_{n+1}$ to consist of the classes of L^M definable from $\mathrm{Tr}_\Gamma^{L^M}(A)$ for some $A \in \mathcal{X}_n$. Then $\mathcal{X} = \bigcup_n$ is a β-ETR_Γ-realization for L^M. □

We also get least transitive models for $\mathsf{ETR}_{\mathrm{Ord}}$ and sufficiently weak fragments of ETR.

Lemma 32. *If transitive $M \models \mathsf{ZFC} + \exists x\ V = \mathrm{HOD}(\{x\})$ is $\mathsf{ETR}_{\mathrm{Ord}}$-realizable then M has a least $\mathsf{ETR}_{\mathrm{Ord}}$-realization.*

Proof. For the proof of lemma 30 the only place it was used that M is β-ETR_Γ-realizable was to get that Γ really is well-founded (i.e. in V). But the Ord of a transitive model of GBC is well-founded, even if it fails to be a β-model. So the exact same argument goes through. That is, the least $\mathsf{ETR}_{\mathrm{Ord}}$-realization for M is $\mathcal{X} = \bigcup_n \mathcal{X}_n$ where $\mathcal{X}_0 = \mathrm{Def}(M)$ and $\mathcal{X}_{n+1} = \bigcup\{\mathrm{Def}(M, \mathrm{Tr}_{\mathrm{Ord}}(A)) : A \in \mathcal{X}_n\}$. □

Theorem 33. *If there is a transitive model of $\mathsf{ETR}_{\mathrm{Ord}}$ then there is a least transitive model of $\mathsf{ETR}_{\mathrm{Ord}}$.*

Proof. Combine lemma 32 plus the fact that M being $\mathsf{ETR}_{\mathrm{Ord}}$-realizable implies L^M is $\mathsf{ETR}_{\mathrm{Ord}}$-realizable. Again, this is proved much the same as lemma 29.1. That is, if $(M, \mathcal{Y}) \models \mathsf{ETR}_{\mathrm{Ord}}$ then $\mathcal{X} = \bigcup_n \mathcal{X}_n$ is an $\mathsf{ETR}_{\mathrm{Ord}}$-realization for L^M, where $\mathcal{X}_0 = \mathrm{Def}(L^M)$ and $\mathcal{X}_{n+1}$ consists of the classes of L^M definable from $\mathrm{Tr}_{\mathrm{Ord}}^{L^M}(A)$ for some $A \in \mathcal{X}_n$. □

The attentive reader will notice that the above argument applies to more than just Ord. Specifically, the only fact about Ord that was used was that if M is transitive then Ord^M is well-founded. The same argument goes through if Ord is replaced by a name for an ordinal or a class well-order which is well-founded in any transitive model. So the same argument goes through more generally, for example if Ord is replaced by ω_1 or $\mathrm{Ord}^{\mathrm{Ord}}$ or $\mathrm{Ord}^{\mathrm{Ord}^{\mathrm{Ord}}} + \mathrm{Ord}^{\mathrm{Ord}} + \mathrm{Ord}$ or many other names for well-orders.

Left open by this analysis is how far we can go. It follows from the proof of theorem 41 that there are names Γ for class well-orders so that there are transitive $(M, \mathcal{X})$ with $\Gamma^{(M,\mathcal{X})}$ ill-founded (i.e. in V). In particular, this holds if Γ is the least class well-order so that L_Γ is admissible. But that definition of Γ requires quantifying over classes. What if we only allow definitions that are first-order?

Question 34. *If Γ is a name for a class well-order defined by a first-order formula, must it be the case that Γ^M is well-founded for any transitive $M \models \mathsf{ZFC}$?*

To finish this section, let us see that ETR is not equivalent to ETR_Γ for any Γ. Moreover, we can separate fragments of ETR. This establishes that theorem 31 is not redundant with theorem 29 and that theorem 33 does not immediately imply that there is a least transitive model of ETR.

First, if Γ is sufficiently smaller than Δ then a model of ETR_Γ need not be a model of ETR_Δ.

Theorem 35. *Let $(M, \mathcal{X}) \models \mathsf{GBC}$ and let $\Gamma \in \mathcal{X}$ be a well-order $\geq \omega^\omega$. Suppose that $(M, \mathcal{X}) \models \mathsf{ETR}_{\Gamma \cdot \omega}$. Then, there is $\mathcal{Y} \subseteq \mathcal{X}$ coded in $\mathcal{X}$ so that $(M, \mathcal{Y}) \models \mathsf{ETR}_\Gamma$ but $(M, \mathcal{Y}) \not\models \mathsf{ETR}_{\Gamma \cdot \omega}$.*

Proof. Fix G a global well-order of M. Define

$$\mathcal{Y} = \{X \in \mathcal{X} : X \in \mathrm{Def}(M, \mathrm{Tr}_\Xi(G)), \text{ where } \Xi \text{ is an initial segment of } \Gamma \cdot \omega^M\}.$$

Then, $\mathcal{Y}$ is coded in $\mathcal{X}$ via the code $C = \{((g, \varphi, a), x) : (g, \varphi, a^\frown x) \in \mathrm{Tr}_{\Gamma \cdot \omega}(G)\}$. Also, $\mathcal{Y}$ is the increasing union of the $\mathrm{Def}(M, \mathrm{Tr}_\Xi(G))$ for Ξ initial segments of $\Gamma \cdot \omega$, each of which is a GBC-realization for M. Thus, $(M, \mathcal{Y}) \models \mathsf{GBC}$. Next, $(M, \mathcal{Y})$ satisfies Elementary Transfinite Recursion for recursions of rank $\leq \Gamma$ because given $X \in \mathcal{Y}$ it must be that $X \in \mathrm{Def}(M, \mathrm{Tr}_\Xi(G))$ for some Ξ an initial segment of $\Gamma \cdot \omega$ and thus $\mathrm{Tr}_\Gamma(X) \in \mathrm{Def}(M, \mathrm{Tr}_{\Xi+\Gamma}(X)) \subseteq \mathcal{Y}$. Finally, $(M, \mathcal{Y}) \not\models \mathsf{ETR}_{\Gamma \cdot \omega}$ because it does not contain $\mathrm{Tr}_{\Gamma \cdot \omega}(G)$. □

This result is optimal for transitive models because ETR_Γ is equivalent to $\mathsf{ETR}_{\Gamma \cdot n}$ for any standard $n \in \omega$.[8]

As an immediate corollary we get that fragments of ETR can be separated by consistency strength.

Corollary 36. *Let Γ be a name for a well-order $\geq \omega^\omega$. Then ETR proves $\mathrm{Con}(\mathsf{ETR}_\Gamma)$. Moreover if Γ and Δ are names for well-orders so that GBC proves $\Gamma \cdot \omega \leq \Delta$ then ETR_Δ proves $\mathrm{Con}(\mathsf{ETR}_\Gamma)$.*

Proof. Let $(M, \mathcal{X}) \models \mathsf{ETR}_\Delta$. By the previous theorem there is $\mathcal{Y} \subseteq \mathcal{X}$ coded in $\mathcal{X}$ so that $(M, \mathcal{Y}) \models \mathsf{ETR}_\Gamma$. But $(M, \mathcal{X})$ has access to the full second-order truth predicate for coded models, via an elementary recursion of length ω. So $(M, \mathcal{X}) \models \mathrm{Con}(\mathsf{ETR}_\Gamma)$. □

In theorem 35 we produced a model $(M, \mathcal{Y})$ of ETR_Γ which was not a model of $\mathsf{ETR}_{\Gamma \cdot \omega}$. But this was due to a deficiency the second-order part. We produced $\mathcal{Y}$ by taking a restricted collection of classes from $\mathcal{X}$ so as to not have long enough iterated truth predicates to get a model of $\mathsf{ETR}_{\Gamma \cdot \omega}$. Can we separate fragments of ETR with the first-order part of a model? That is, is there $M \models \mathsf{ZFC}$ so that M is ETR_Γ-realizable but not $\mathsf{ETR}_{\Gamma \cdot \omega}$-realizable?

Of course, if we put no restriction on M then this follows immediately from the fact that $\mathsf{ETR}_{\Gamma \cdot \omega}$ proves the consistency of ETR_Γ. If M is the first-order part of a model of $\mathsf{ETR}_\Gamma + \neg\,\mathrm{Con}(\mathsf{ETR}_\Gamma)$ then M will not be $\mathsf{ETR}_{\Gamma \cdot \omega}$-realizable. Such M will necessarily be ω-nonstandard. Can we make the same separation, but with an M which is ω-standard, or even transitive?

The answer is yes, looking below $\mathsf{ETR}_{\mathrm{Ord}}$.

[8]For ω-nonstandard models ETR_Γ is not equivalent to $\mathsf{ETR}_{<\Gamma \cdot \omega}$. Suppose $(M, \mathcal{X}) \models \mathsf{ETR}_{\Gamma \cdot \omega}$ and let $I \subseteq \omega^M$ be an initial segment closed under addition. A similar construction can be used to get $\mathcal{Y} \subseteq \mathcal{X}$ so that $(M, \mathcal{Y})$ satisfies $\mathsf{ETR}_{\Gamma \cdot n}$ if and only if $n \in I$.

Theorem 37. *Suppose γ is a name for an ordinal $\geq \omega^\omega$. (In particular, γ is a set, not a proper class.) If there is a transitive model of $\mathsf{ETR}_{\gamma\cdot\omega}$ then there is transitive M which is ETR_γ-realizable but not $\mathsf{ETR}_{\gamma\cdot\omega}$-realizable.*

Proof. Take $(N, \mathcal{Y}) \models \mathsf{ETR}_{\gamma\cdot\omega}$ which has a first-order definable global well-order. Consider the iterated truth predicates $\vec{T} = \{\mathrm{Tr}_\xi : \xi = \gamma \cdot n\}$ and let $(M, \vec{S}) = \mathrm{Sk}^{(N,\vec{T})}(\gamma)$ be the Skolem hull of γ in the structure $(M, \vec{T})$. By elementarity, $\vec{S} = \{(\mathrm{Tr}_\xi)^M : \xi = \gamma \cdot n\}$. Moreover, $\gamma \cdot n \in N$ for all n because $(N, \vec{T})$ satisfies that the length of each truth predicate in $\vec{T}$ is a set.

Let $\mathcal{X} = \mathrm{Def}(M, \vec{S})$. Then, $(M, \mathcal{X}) \models \mathsf{ETR}_\gamma$; given $A \in \mathcal{X}$ there is n so that A is definable from $\mathrm{Tr}_{\gamma\cdot n}(\bar{G})$. But then $\mathrm{Tr}_\gamma(A)$ is definable from $\mathrm{Tr}_{\gamma\cdot(n+1)}$.

However, M is not $\mathsf{ETR}_{\gamma\cdot\omega}$-realizable. This is because the $(\gamma \cdot \omega + 1)$-iterated truth predicate for M reveals that M is a Skolem hull and hence countable. But no model of ZFC thinks that it is countable. So no GBC-realization for M can contain $\mathrm{Tr}^M_{\gamma\cdot\omega+1}$, which $\mathsf{ETR}_{\gamma\cdot\omega}$ proves exists. □

Left open by this section is whether ETR has a least transitive model.

The proof that there is a least β-model of ETR came down to two facts: first, that if a model of ZFC with a definable global well-order is β-ETR-realizable then it has a least β-ETR-realization; second, that if M is β-ETR-realizable then so is L^M. One strategy to try to prove the existence of a least transitive model of ETR would be to prove analogues of these two facts. The latter—i.e. that if M is ETR-realizable then so is L^M—can be proven by an argument similar to the proof of lemma 29.1, but done entirely interally to $(M, \mathcal{X})$.

I do not see how to settle the analogue of the first fact. Working towards a positive answer, the proof of theorem 29 does not generalize to show that transitive ETR-realizable models with definable global well-orders have least ETR-realizations. Working towards a negative naswer, the argument in the next section that no countable model of ZFC has a least KM-realization does not generalize down to ETR. The trouble is that ETR is too weak for the same proof to go through. Namely, ETR does not prove that there is an admissible "meta-ordinal".

Question 38. *If M is ETR-realizable and has a definable global well-order, must M have a least ETR-realization? What if M is transitive?*

A positive answer to this question would imply a positve answer to the following, the main open question from this article.

Question 39. *Is there a least transitive model of ETR?*

6. KM

First let us prove the folklore result that there is a smallest β-model of KM.

Theorem 40. *There is a least β-model of KM, if there is any β-model of KM.*

Proof. Suppose there is a β-model of KM. By theorem 12 there is a β-model of KM^+. Its unrolling is a well-founded model of $\mathsf{ZFC}^-_\mathsf{I}$, so there is a transitive model of $\mathsf{ZFC}^-_\mathsf{I}$. Let L_α be the least transitive model of $\mathsf{ZFC}^-_\mathsf{I}$.[9] Let $\kappa < \alpha$ be the largest cardinal in L_α. Set $(M, \mathcal{X}) = (L_\kappa, \mathcal{P}(L_\kappa) \cap L_\alpha)$. I claim that $(M, \mathcal{X})$ is the least β-model of KM. First, observe that it is itself a β-model since L_α is correct about well-foundedness.

[9]To see such exists: Take $M \models \mathsf{ZFC}^-_\mathsf{I}$ transitive with $\kappa \in M$ the largest cardinal. Then κ is inaccessible in L^M so $L_{(\kappa^+)^{L^M}} \models \mathsf{ZFC}^-_\mathsf{I}$. So the least transitive model of $\mathsf{ZFC}^-_\mathsf{I}$ is L_α for α least such that there is a transitive model of $\mathsf{ZFC}^-_\mathsf{I}$ of height α.

Take $(N, \bar{\mathcal{Y}})$ a β-model of KM. By theorem 12 there is $\mathcal{Y} \subseteq \mathcal{Y}$ so that $(N, \mathcal{Y}) \models \mathsf{KM}^+$. Now let $W \models \mathsf{ZFC}^-_\mathsf{I}$ be the unrolling of $(N, \mathcal{Y})$. Because $(N, \mathcal{Y})$ is a β-model, W must be well-founded. Without loss assume that W is transitive. Then, $L_\alpha \subseteq W$.

Let λ be the largest cardinal in W. There are two cases. The first is if $\lambda > \kappa$. Then, because W thinks λ is inaccessible, W thinks $\mathcal{P}(L_\kappa)$ exists. Therefore, $\mathcal{X} = \mathcal{P}(L_\kappa) \cap L_\alpha \in V^W_\lambda = N$. The second case is if $\lambda = \kappa$. Then, $\mathrm{Ord}^W \geq \alpha$ by the minimality of L_α. Thus, $\mathcal{X} = \mathcal{P}(L_\kappa) \cap L_\alpha \subseteq \mathcal{P}(L_\kappa) \cap W$ so $\mathcal{X} \subseteq \mathcal{Y}$. □

The same is not true if we look at transitive models, rather than confining ourselves to β-models.

Theorem 41 (Main Theorem 1)**.** *There is no least transitive model of* KM*. Moreover, for any real r there is no least transitive model of* KM *which contains r.*

Note that it suffices to prove the moreover, as $r = \emptyset$ gives the other result.

Proof of theorem 41. Suppose otherwise towards a contradiction that $(M, \mathcal{X})$ is the least transitive model of KM which contains r. By a Löwenheim–Skolem argument it must be that M and $\mathcal{X}$ are countable. Moreover, by theorem 12, $(M, \mathcal{X})$ must be a model of KM^+. Let $W \models \mathsf{ZFC}^-_\mathsf{I}$ be the unrolling of $(M, \mathcal{X})$. By taking an isomorphic copy, if necessary, of $(M, \mathcal{X})$, assume without loss that $M \subseteq W$ and $\mathcal{X} = \{X \in W : W \models X \subseteq M\}$. Let κ be the largest cardinal in W. In particular, $M = V^W_\kappa$.

First, we want to find $A \in W$ so that W believes A is an admissible set with M as an element. This is done via the standard argument for the Montague reflection principle, except that we cannot use the V_ξ hierarchy because the collection of sets of rank $\xi \geq \kappa$ will not be a set in W. Instead fix a bijection $G : M \to \kappa$ and we will work with the $L_\xi(M, G)$ hierarchy.

Lemma 41.1. *Work inside a model of* ZFC^- *and let φ be a formula. Then for any set m there is a set $a \ni m$ so that $a \models \varphi$ if and only if $\varphi^{L(m)}$.*

Proof. Define a sequence $\langle \xi_n : n \in \omega \rangle$ as follows. Start with arbitrary ξ_0 so that $m \in L_{\xi_0}(m)$. Given ξ_n let ξ_{n+1} be the least ordinal α so that $L_\alpha(m)$ has witnesses for existential subformulae of φ with parameters from $L_{\xi_n}(m)$, if such witnesses exist in W. Such α exists by an instance of Collection. Finally, set $\xi = \sup_n \xi_n$. Then $L_\xi(m) \models \varphi$ iff $\varphi^{L(m)}$. Set $a = L_\xi(m)$. □

Let φ be a formula so that $A \models \varphi$ if and only if $A \models \mathsf{KP}$. This can be expressed as a single formula via the usual construction of bounded truth predicates. By the lemma find $A \ni M, G$, where G is the bijection $\kappa \to M$ fixed above, in W so that $A \models \mathsf{KP}$. Such exists because $L(M, G) \models \mathsf{ZFC}^-$ and thus $L(M, G) \models \mathsf{KP}$.

We are now in a position to apply theorem 15, the generalization of Friedman's theorem. Consider the theory which consists of the following assertions.

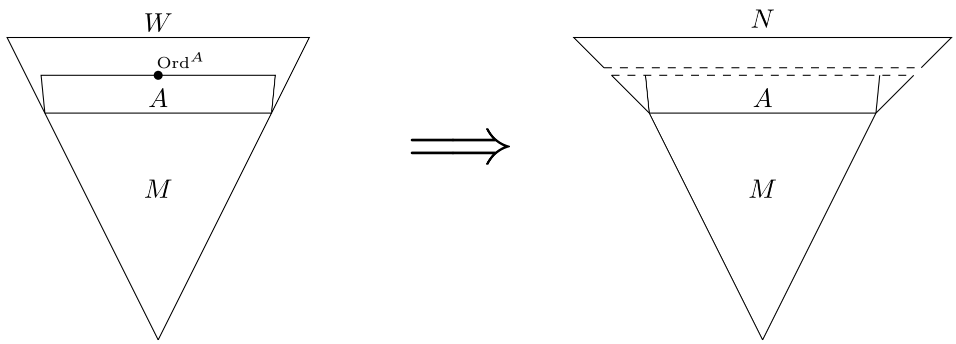
- All axioms of $\mathsf{ZFC}^-_\mathsf{I}$;
- The assertion that M is a rank-initial segment of the universe; and
- The assertion that $\kappa = \mathrm{Ord}^M$ is the largest cardinal and is inaccessible.

Note that W is a model of this theory which contains A.

Let $\mathfrak{C} = \mathrm{Cov}_A$ be the admissible cover of A (see section 3 for definitions). Because M is an element of A, this theory can be axiomatized by a single $L_\mathfrak{C}$-formula and is *a fortiori* Σ_1-definable in $\mathfrak{C}$. Applying theorem 15 to A and this theory we get $N \models \mathsf{ZFC}^-_\mathsf{I}$ such that

- N is an end extension of A;
- Ord^A is a proper initial segment of Ord^N;

- A is not topped in N—there is no $\alpha \in \mathrm{Ord}^N$ so that $\{\beta \in \mathrm{Ord}^N : N \models \beta < \alpha\} = \mathrm{Ord}^A$; and
- $M = V_\kappa^N$ is a rank-initial segment of N, where κ is the largest cardinal of N.



Friedman's theorem gives $N \models \mathsf{ZFC}^-_\mathsf{I}$ with $V_\kappa^N = M = V_\kappa^W$ and N ill-founded at A.

Let $\mathcal{Y} = \{Y \in N : N \models Y \subseteq M\}$. Then $(M, \mathcal{Y}) \models \mathsf{KM}^+$. If $(M, \mathcal{X})$ is a β-model (or just $\mathrm{Ord}^A < \mathrm{Ord}^{\mathrm{wfp}(W)}$), then Ord^A really is an ordinal. Thus, $\mathcal{Y}$ has no element with ordertype Ord^A because otherwise Ord^A would be in the well-founded part of N, contrary to the construction. This contradicts the leastness of $\mathcal{X}$.

It might be, however, that $\mathcal{X}$ is wrong about well-foundedness.[10] One could imagine that $\mathcal{X}$ came from applying Friedman's theorem as above to a β-model. If this were done using the least admissible set with M as an element then every meta-ordinal which $\mathcal{X}$ thinks is the height of an admissible class with M as an element would be ill-founded. The argument from the previous paragraph does not work in this case. Then, $\mathcal{Y}$ will have meta-ordinals it thinks are non-isomorphic but which are seen from the ambient universe to be isomorphic to Ord^A. This is a consequence of a theorem, due to H. Friedman [Fri73], that if M is a countable ill-founded ω-model of KP that $\mathrm{Ord}^M \cong \alpha + \alpha \cdot \mathbb{Q}$, where $\alpha = \mathrm{Ord}^{\mathrm{wfp}(M)}$.

Nevertheless, even if A is ill-founded, $\mathcal{Y}$ will not contain an element representing Ord^A in the unrolled model N. This will contradict the assumed leastness of $\mathcal{X}$, as desired.

Let us see why this is. Pick $\Delta \in \mathcal{X}$ so that $W \models \Delta \cong \mathrm{Ord}^A$. That is, Δ represents Ord^A in the unrolling. By the minimality of $\mathcal{X}$ we have $\Delta \in \mathcal{Y}$. Because being a well-order is an elementary property, $(M, \mathcal{Y}) \models \Delta$ is a well-order. Let $\delta \in N$ be an ordinal so that $N \models \delta \cong \Delta$.

I claim for any $\gamma \in \mathrm{Ord}^N$ that $N \models \gamma < \delta$ if and only if $\gamma \in A$. The backward direction is clear, since $\gamma \in A$ implies there is $\pi \in \mathcal{X}$ so that π is a map witnessing that $\Gamma \cong \gamma$ is an initial segment of Δ and the leastness of $\mathcal{X}$ implies $\pi \in \mathcal{Y}$. For the forward direction, suppose $N \models \gamma < \delta$. Then there is $g \in \operatorname{dom} \Delta$ so that $N \models \gamma \cong \Delta \upharpoonright g = \{(d, d') \in \Delta : d <_\Delta d' <_\Delta g\}$. In $\mathcal{X}$ we have $\Delta \upharpoonright g$ must represent an ordinal $< \delta$ and thus there is $\Gamma \in A \cap \mathcal{X}$ so that Γ represents γ. But then $\gamma \in A$.

So Ord^A is topped in N by δ, contradicting the construction of N. Thus our original assumption that there is a least transitive model of KM must be false. □

Corollary 42. *No countable model of set theory has a least* KM*-realization.*

[10] Indeed, if M is the least transitive KM-realizable model then it must be that $\mathcal{X}$ is wrong about well-foundedness. See [MM75, theorem 3.2].

Proof. This follows from the proof of the previous theorem. Nowhere in the proof did we use the fact that M was transitive.

Suppose that $\mathcal{X}$ were the least KM-realization for countable $M \models$ ZFC. As before, it must be that $\mathcal{X}$ is a countable KM^+-realization for M. The same argument gives a KM^+-realization $\mathcal{Y}$ for M so that $\mathcal{X} \not\subseteq \mathcal{Y}$, contradicting the leastness of $\mathcal{X}$. □

We can also extract from the proof of theorem 41 the following well-known result.

Corollary 43. *There are transitive models of* KM *which are not β-models.* □

(Indeed, there are transitive KM-realizable models which have no β-KM-realization; Marek and Mostowski [MM75] proved that the least height of a transitive KM-realizable model is less than the least height of a β-KM-realizable model.)

As in the GBC case, this result leaves open the question of minimal but non-least KM-realizations. Also open is whether there is a transitive model of KM which is minimal—but necessarily not least—among the transitive models of KM.

Question 44.

- *Which (countable) models of* ZFC *have minimal* KM*-realizations?*
- *Is there a minimal transitive model of* KM*? That is, does there exist transitive* $(M, \mathcal{X}) \models$ KM *so that if* $(N, \mathcal{Y}) \models$ KM *is contained inside* $(M, \mathcal{X})$ *then* $N = M$ *and* $\mathcal{Y} = \mathcal{X}$*?*

The first question has an easy positive answer for a certain natural case.

Corollary 45. *If M has a least β-*KM*-realization then it has a minimal (but non-least)* KM*-realization*

Proof. Let $\mathcal{X}$ be the least β-KM-realization for M. By observation 18, any $\mathcal{Y} \subseteq \mathcal{X}$ must also be a β-realization. So if $\mathcal{Y} \subseteq \mathcal{X}$ is a KM-realization then $\mathcal{Y} = \mathcal{X}$. So $\mathcal{X}$ is a minimal KM-realization for M. □

The general case remains open.

6.1. **Further and prior results.** As a last matter I wish to briefly explicate how main theorem 1 can be generalized and then to contrast the result with previous theorems in the literature.

First, generalizing the main theorem: The key fact used in the proof was that KM^+ is bi-interpretable with a first-order theory which proves that $\mathrm{Hyp}(X)$, the smallest admissible set with X as an element, exists for every set X. The Marek–Mostowski argument can be generalized to get that models of ETR can be unrolled into first-order models. Moreover, Π^1_1-CA suffices to get that the unrolled model has that $\mathrm{Hyp}(X)$ exists for all sets X. As a consequence one can obtain the following, a full proof of which will appear in my forthcoming dissertation.

Theorem 46. *Let $k \geq 1$. Then there is not a least transitive model of* Π^1_k-CA.

See also [ABF, theorem 64], where Antos, Barton, and S. Friedman show that Π^1_1-CA suffices to prove that $\mathrm{Hyp}(M)$, where M is the first-order part, is coded as a single class.

Second, H. Friedman [Fri73] showed that there is not a least ω-model of Z_2, full second-order arithmetic. Nowhere in the proof of theorem 41 was it used that the KM model satisfies the axiom of Infinity. Because the Marek–Mostowski bi-interpretability result does not need the axiom of

Infinity, this gives that there is no least transitive model of $\mathsf{KM}^{\neg\infty}$.[11] Using that $\mathsf{KM}^{\neg\infty}$ and Z_2 are bi-interpretable this gives a new proof that there is no smallest ω-model of Z_2.

Finally, I wish to briefly contrast my results with two previous results in the literature about least models of KM. There are subtleties which are obscured by the language used and I wish to clarify how this article fits into the broader context.

The first is a result due to Marek and Mostowski [MM75]. In their language, they proved that there are minimal models of KM. Let $\mathsf{KM}^{\neg\mathsf{GC}}$ denote KM with Global Choice thrown out. Marek and Mostowski proved, in my language, that every β-$\mathsf{KM}^{\neg\mathsf{GC}}$-realizable model has a least β-$\mathsf{KM}^{\neg\mathsf{GC}}$-realization. As an immediate corollary, if M has a definable global well-order and is β-KM-realizable then it has a least β-KM-realization. See also my corollary 45.

The second result is due to Antos and S. Friedman [AF]. In their language, they proved that every β-model of KM^+ can be extended to a minimal β-model of KM^+. For them, a minimal β-model of KM^+ is a β-model M of KM^+ so that there is some real S so that any β-model of KM^+ which contains S must contain M. My theorem 41 shows that their result cannot possibly be generalized from β-models to transitive models. I see this as justifying Antos and Friedman's restriction to looking only at β-models. They justify this restriction (in their footnote 3) by analogy to ZFC, saying that it only makes sense to talk about minimal models for well-founded models. I view this analogy as unfounded; there is a least transitive model of GBC—and even of $\mathsf{ETR}_{\mathrm{Ord}}$, which is far stronger than GBC—least among even the models which are wrong about well-foundedness. Nevertheless, their restriction is necessary to get the sort of result they are looking at.

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[11] Here, $\mathsf{KM}^{\neg\infty}$ is KM with the axiom of Infinity thrown out. However, this is sensitive to how one formulates Foundation, as different formulations are not equivalent in the absence of Infinity. So $\mathsf{KM}^{\neg\infty}$ should be understood as including the correct formulation of Foundation. See Kaye and Wong's discussion [KW07] of the similar situation of $\mathsf{ZFC}^{\neg\infty}$.

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